Int. J. Heat Mass Transfer. Vol. 15, pp. 35–42. Pergamon Press 1972. Printed in Great Britain

HEAT TRANSFER IN THE ENTRANCE REGION WITH FULLY DEVELOPED TURBULENT FLOW BETWEEN PARALLEL PLATES

B. LECKNER

Department of Steam Engineering, Chalmers University of Technology, 40220 Göteborg 5, Sweden

(*Received* 5 *November* 1970 *and in revised form* 5 *April* 1971)

Abstract – An analysis is made of local heat transfer in a thermal entrance region with a fully developed turbulent velocity profile for various Reynolds and Prandtl numbers. Difference methods are used to solve the energy equation. The velocity and eddy diffusivities are described by semi-empirical relationships. The geometrical picture is simple: flow between parallel plates, one of which is thermally insulated, the other transmits a known heat flux. Certain comparisons with calculations on laminar flow are made as a check on the reliability of the numerical method. This also provides an estimate of the influence of turbulence in the various cases and gives a possibility of judging the validity of approximations. Finally a comparison with available empirical expressions for fully developed heat transfer is shown.

NOMENCLATURE

$2d$,	width of the channel;
c_p,	specific heat;
$f(\ )$,	function;
q,	heat flux;
t,	temperature;
u,	velocity;
$\bar{u}$,	mean velocity;
x, y,	coordinates;
α,	heat transfer coefficient;
ε_H,	eddy diffusivity for heat;
ε_M,	eddy diffusivity for momentum;
ρ,	density;
λ,	fluid thermal conductivity;
τ,	shear stress;
ν,	kinematic viscosity.

Dimensionless numbers

$Nu = 4d\alpha/\lambda$,	Nusselt number;
$Pr = \nu c_p \rho/\lambda$,	Prandtl number;
$Re = 4d\bar{u}/\nu$,	Reynolds number;
$Pe = Re.Pr$,	Peclet number;
$u^+ = u/\sqrt{(\tau_w/\rho)}$,	velocity in the universal profile;
$y^+ = y\sqrt{(\tau_w/\rho)}/\nu$,	distance from the wall in the universal profile;
$T = t/t_0$,	temperature;
$U = u/\bar{u}$,	velocity;
$X = x/d$,	distance from the beginning of the thermal entrance region;
$Y = y/d$,	distance from the wall;
$\Delta X, \Delta Y, \Delta Y^+$,	step lengths.

Subscripts

x,	defines location;
b,	mean;
0,	initial condition;
w,	wall;
∞,	fully developed condition;

INTRODUCTION

THE PURPOSE of this calculation is to give a description of the heat transfer in a thermal entrance region. The fluid, which has temperature-independent properties, flows with a fully developed turbulent velocity profile between parallel plates. One of the plates is thermally insulated and the other transmits a known heat flux. A standard implicit difference method [1] is used to solve the energy equation. The calculation is primarily intended for fluids with low Prandtl numbers, i.e. liquid metals, but the simple problem studied allows examination of the thermal entrance region over a wide range of Reynolds and Prandtl numbers including extremely small distances from the beginning of the entrance region, as well as fully developed heat transfer far downstream.

ANALYSIS

The heat transfer section starts at $x = 0$ and the channel width is $2d$ (Fig. 1). Boundary conditions: No heat transfer $q = 0$ at the wall $y = 2d$ and a known heat transfer rate q at the wall $y = 0$.

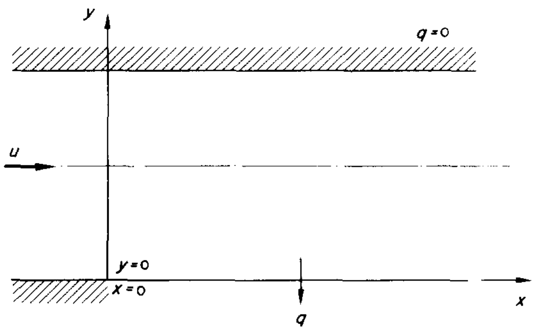

FIG. 1. Coordinate system.

Entrance conditions: Fully developed turbulent flow with a velocity $u = u(y)$ and a constant temperature $t = t_0$ at $x = 0$.

Additional assumptions: The Reynolds numbers are sufficiently large to permit neglect of natural convection and heat transport in the direction of the flow for all fluids considered and in all parts of the channel. Then, the velocity is sufficiently well approximated by the universal velocity profile and the wall shear stress τ_w with the empirical relations

$$\tau_w = 0{\cdot}046\, Re^{-0{\cdot}2} \frac{\bar{u}^2}{2} \tag{1}$$

and for $Re < 50000$

$$\tau_w = 0{\cdot}079\, Re^{-0{\cdot}25} \frac{\bar{u}^2}{2} \tag{2}$$

where $\bar{u}$ is the mean velocity and $Re = \frac{4d\bar{u}}{\nu}$.

Further, the eddy diffusivity of heat ε_H is assumed to be equal to the eddy diffusivity of momentum ε_M. The latter is described by means of van Driest's model [2] with slight modifications due to the pressure drop in the channel. Thus, with $u^+ = u/\sqrt{\tau_w/\rho}$ and $y^+\sqrt{\tau_w/\rho}/\nu$

$$\frac{du^+}{dy^+} = \frac{2(1 - y/d)}{1 + \sqrt{\{1 + 0{\cdot}64 y^{+2} \times [1 - \exp(-y^+/26)](1 - y/d)\}}} \tag{3}$$

is assumed to apply for $y^+ < 30$ and the derivative of the universal velocity profile

$$\frac{du^+}{dy^+} = \frac{2{\cdot}5}{y^+} \tag{4}$$

until $y/d = 0{\cdot}8$. In the central part of the channel $|1 - y/d| < 0{\cdot}2$ ε_M is kept constant and equal to the value at $y/d = 0{\cdot}8$. This is arbitrarily chosen in order to avoid the less probable value of zero at $y/d = 1$. The velocity—and ε_H—profiles are duplicated in the other part of the channel, $1 < y/d < 2$. To indicate the sensitivity on the heat transfer results of the choice of eddy diffusivity relation, the equation (3) is used in the whole channel except in the central part $|1 - y/d| < 0{\cdot}2$ in some of the calculations. This means an increase of ε_H at about 50 per cent in certain central regions of the passage.

Solution of the energy equation

$$U\frac{\partial T}{\partial X} = \frac{\partial}{\partial Y}\left[\frac{4}{RePr} + \frac{4\varepsilon_H}{Re\,\nu}\right]\frac{\partial T}{\partial Y} \tag{5}$$

expressed in the dimensionless quantities $X = x/d$, $Y = y/d$, $T = t/t_0$, $U = u/\bar{u}$ and $Pr = \nu c_p \rho/\lambda$ gives the temperature field and thus the local Nusselt number Nu_x

$$Nu_x = \frac{4(\partial T/\partial Y)_w}{T_{b,x} - T_{w,x}} \tag{6}$$

where subscript w indicates properties at the wall $Y = 0$. The mean temperature is given by

$$T_{b,x} = \tfrac{1}{2}\int_0^2 T \,.\, U \,.\, dY. \tag{7}$$

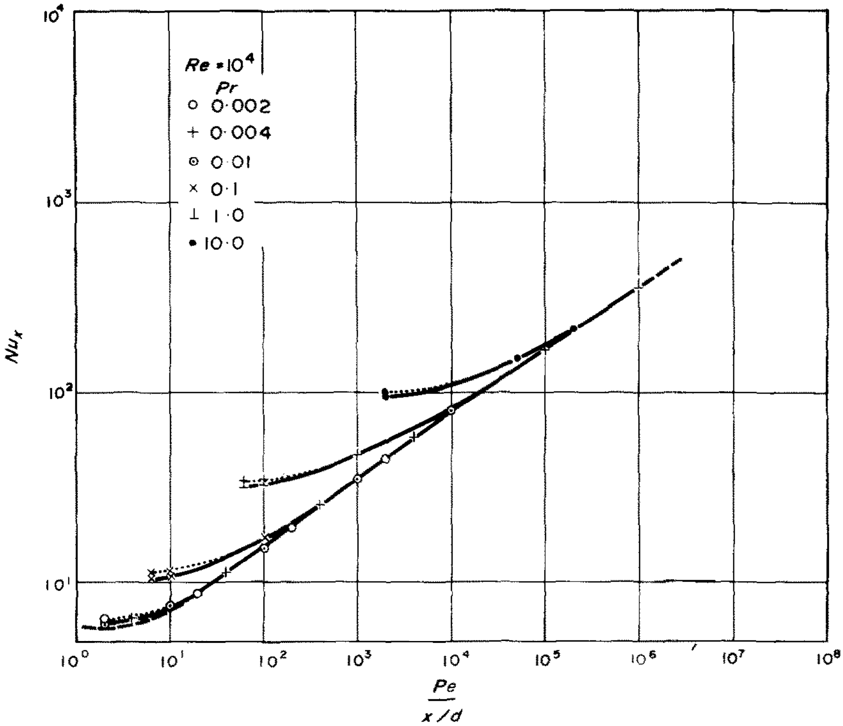

FIG. 2a. $Nu_x = f[Pe/(x/d)]$ at $Re = 10000$. Dotted lines—ε_M/ν according to equation (3). Dashed lines—$\varepsilon_M/\nu = 0$.

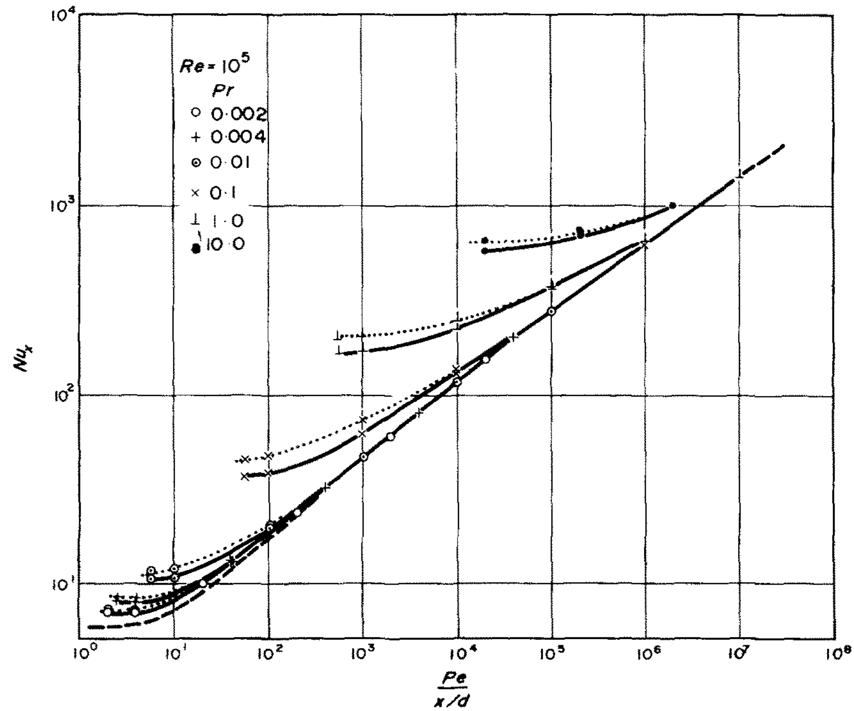

FIG. 2b. $Nu_x = f[Pe/(x/d)]$ at $Re = 100000$. Dotted lines—ε_M/ν according to equation (3). Dashed lines—$\varepsilon_M/\nu = 0$.

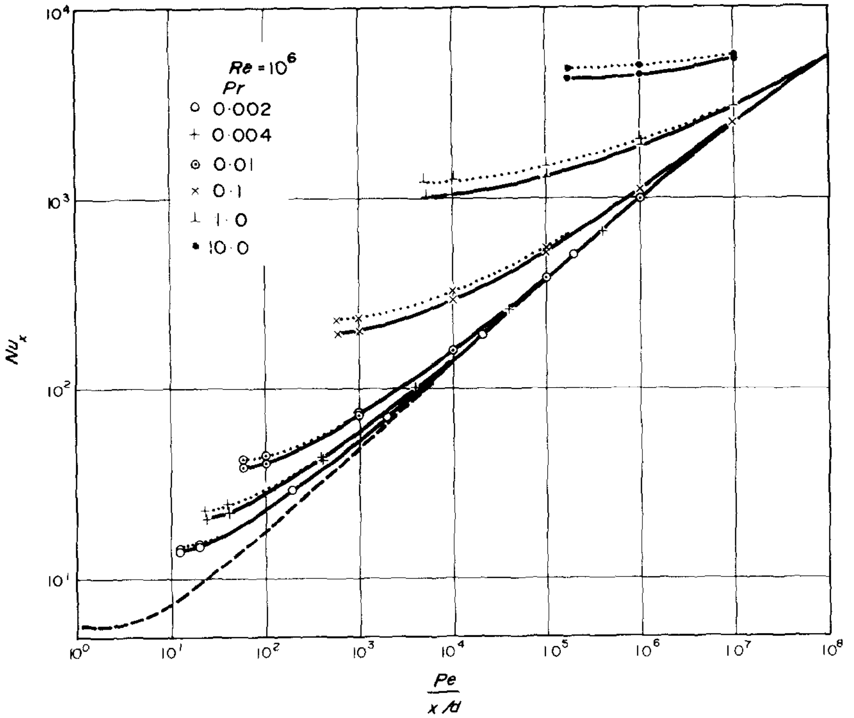

FIG. 2c. $Nu_x = f[Pe/(x/d)]$ at Re = 1000000. Dotted lines—ε_M/ν according to equation (3). Dashed lines—$\varepsilon_M/\nu = 0$.

The difference solution of equation (5) has been performed with the step lengths ΔX = 0·01, 0·1 and 1·0 in the x-direction, starting from $x = 0$. The step length in the y-direction, starting from $y = 0$ has been subjected to the condition that at least one node should be placed in the viscous sublayer. In effect, the first step length, expressed in the coordinates of the velocity profile, has been chosen $\Delta y_1^+ = \frac{5}{4}$ or $\Delta y_1^+ = \frac{5}{10}$. The following step lengths are successively increased until the step length has reached a certain maximum value.

RESULTS

The results for some combinations of Re and Pr are presented in the form $Nu_x = f(Pe/x/d)$ with Pr or Pe as a parameter in Figs. 2 a–c. The case $Re = 10^4$ is an extreme case since measured velocities starts to deviate somewhat from the universal velocity profile at such low Re [3]. The curves for each value of the parameter go with increasing $Pe/x/d$ from the area of fully developed heat transfer towards an asymptote, which is a straight line (in the log–log scale) over a large $Pe/x/d$ – interval and depends on Reynolds number, that is, on the velocity profile. The asymptotes deviate from the straight lines close to $x = 0$, where the thermal boundary layer is entirely or to a large extent within the viscous sublayer.

If the calculation is repeated with the same turbulent velocity profiles as above, but with the contribution from the turbulent heat transport omitted ($\varepsilon_H = 0$) the asymptotes are obtained in their full length (the dashed lines in Figs. 2 a–c and Fig. 3). The deviation from these asymptotes is consequently due to the influence of turbulent heat transport on the heat transfer. The dotted curves are those calculated with

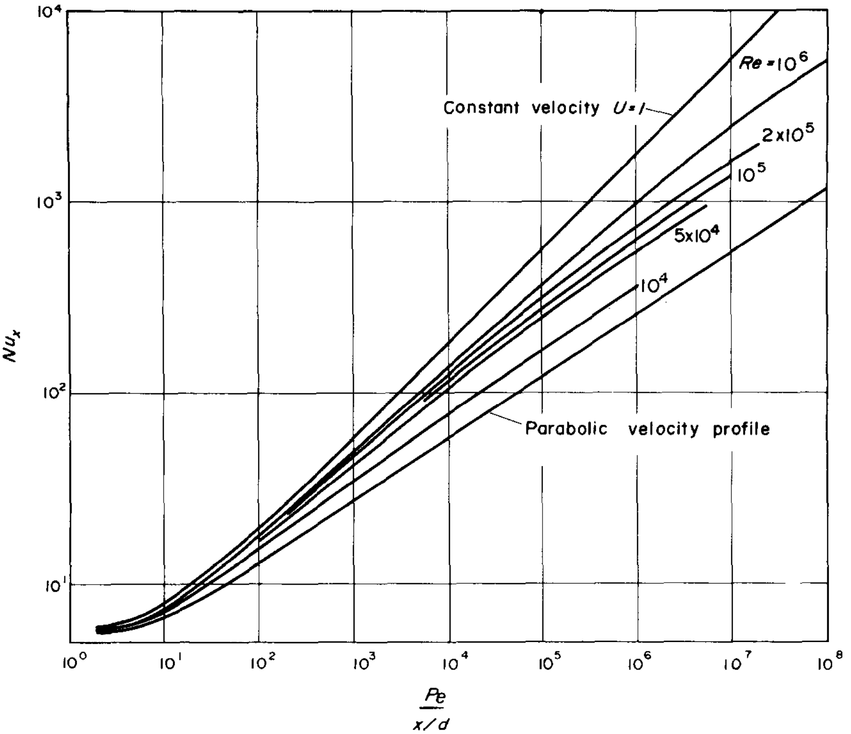

FIG. 3. The asymptotes for various turbulent velocity profiles and the limiting cases, constant velocity and parabolical velocity profile. ($\varepsilon_M/\nu = 0$ in all cases.)

equation (3) in the whole passage except $|1 - y/d| < 0{\cdot}2$.

The higher the Reynolds number, the more the velocity profile approaches the case of constant velocity over the cross-section ($U = 1$) and at lower Re it looks more like a parabolical profile. A comparison with these limiting cases (both with $\varepsilon_H = 0$) is illustrated in Fig 3. The parts of the asymptotes that deviate from the straight line at large $Pe/x/d$ reach ultimately a slope corresponding to that of the parabolic limiting case. It seems, thus, to be possible to extrapolate the curves to $x = 0$.

It is often assumed that the relative heat transfer Nu_x/Nu_∞ is larger in liquid metals than in other fluids with higher Pr. Recently, Chen and Yu [4] have indicated that this might not be the case. Figure 4 shows that maximum relative heat transfer at $x/d = 1$ occurs at rather high Prandtl numbers for low Reynolds numbers and that the maximum moves towards lower Prandtl numbers when Reynolds numbers increases. From the same figure it is seen that there is a reversal of the Reynolds number

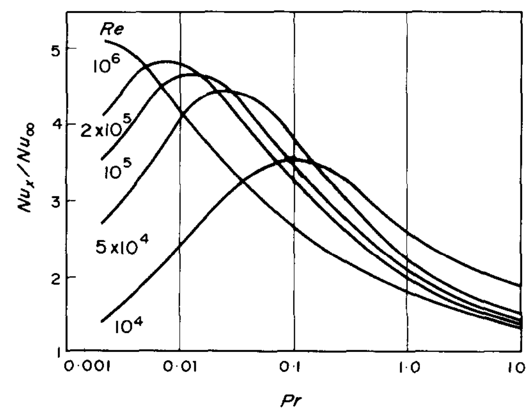

FIG. 4. The relative heat transfer at $x/d = 1$ as a function of Pr at various Re.

dependency for low and high Prandtl numbers. (The curves in Fig. 4 are interpolated between calculated points at the *Pr* given in Fig. 2. The same tendencies also occur at $x/d < 1$ and with some exceptions for $x/d > 1$.)

ESTIMATION OF ERRORS

It is evident that the numerical method is affected by a "starting effect" that appears as a calculated Nu_x in the first step in the x-direction that is about 10 per cent too high (this occurs at all step lengths $\Delta X = 0{\cdot}01$, $0{\cdot}1$ and $1{\cdot}0$). This starting effect diminishes rapidly and is negligible after the 3–5th step. The effect is distinguishable and may be corrected for, as the three step lengths all start from $X = 0$ and the results overlap each other on a large part of the x/d axes if diagrams are plotted in the form of $Nu_x = f(x/d)$. The fact that the curves for different *Pr* coincide with the asymptotes indicates that the correction has been performed properly, the only restriction being that it has been done in the form of straight lines, which is seen in Figs. 2 and 3.

The step length in the *y*-direction has an importance at small x/d and large *Pr* where the thermal boundary layer is narrow. Initial step lengths, half as small as the above mentioned did not give any noticeable change in the results, but the initial step length $\Delta y_1^+ = \frac{5}{2}$ made Nu_x 1–2 per cent lower at small x/d and large *Pr*.

In order to check the accuracy of the calculations a comparison can be made with other calculations or with empirical relationships as far as such are available. Hatton and Quarmby's calculation [5] covers a limited part of the transition region between the entrance section and the fully developed heat transfer section for some *Re* and *Pr*. Their results agree with those presented here within the accuracy of the eddy diffusivity assumptions. Poppendiek's calculation [6] for liquid metals is valid for the boundary condition of $t_w =$ const. and heat transfer at both plates. It consequently gives heat transfer results, which after a similarity transformation, are seen to be somewhat lower than those of the present calculation in the transition region. The approximation of constant

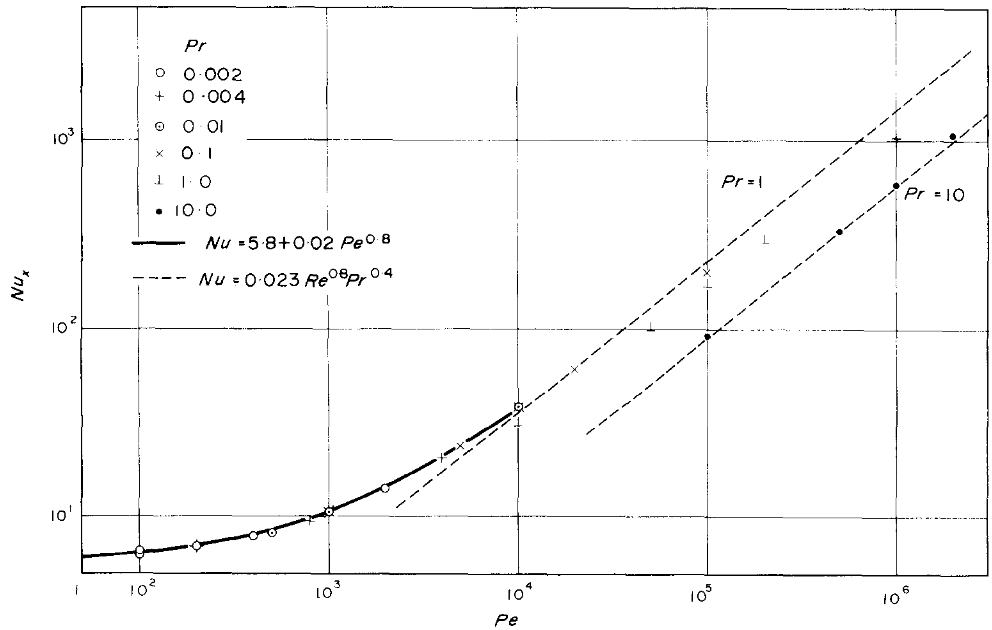

FIG. 5. Fully developed heat transfer. Comparison between calculated points and empirical relationships.

velocity over the cross-section makes Poppendiek's curves approach an asymptote corresponding to $(Pr/x/d)^{\frac{1}{2}}$. By means of Fig. 3 the error of such an approximation in calculating Nu_x can be estimated.

Seban's relation [7], valid for fully developed heat transfer from liquid metals flowing between parallel plates and the Dittus–Boelter relation used for fluids with $Pr \geqslant 0.7$ are compared with the values calculated with the turbulence model equations (3) and (4) in Fig. 5. If equation (3) only is used the calculated Nu_∞ would fall insignificantly above the curve representing Seban's relation and agree better with the dashed line $Pr = 1$, but fall somewhat above the dashed line $Pr = 10$.

DISCUSSION

Excluding the starting effect, which has been corrected for, the numerical treatment does not seem to have caused errors of such a magnitude that they are visible in the plotting of the diagrams.

The only divergence that may be pointed out with certainty is that of the value of Nu_∞ at $Pr \sim 1$ and larger. The reason of this is most probably to be found in the description of the turbulent heat transport. The same tendency may be found in analytical calculations when the assumption $\varepsilon_H/\varepsilon_M = 1$ is used, e.g. [3, 5]. The remedy usually employed is to put $\varepsilon_H/\varepsilon_M \approx 1.2$ over the whole passage or over parts of it, e.g. [8]. This leads to better agreement at $Pr \approx 1$ but to larger error at $Pr \approx 10$. The turbulent transport assumptions have of course less importance in calculations with fluids of low Pr with dominant molecular heat transport. This can be seen in Fig. 2.

REFERENCES

1. R. D. RICHTMEYER and K. W. MORTON, *Difference Methods for Initial Value Problems* Interscience Publishers, New York (1967).
2. E. R. VAN DRIEST, On turbulent flow near a wall, *J. Aeronaut. Soc.* **23**, 1007–1011 (1956).
3. C. J. LAWN, Turbulent heat transfer at low Reynolds numbers, *J. Heat Transfer* **91**, 532–536 (1969).
4. J. C. CHEN and W. S. YU, Entrance region and variable heat flux effects in turbulent heat transfer to liquid metals flowing in concentric annuli, *Int. J. Heat Mass Transfer* **13**, 667–680 (1970).
5. A. P. HATTON and A. QUARMBY, The effect of axially varying and unsymmetrical boundary conditions on heat transfer with turbulent flow between parallel plates, *Int. J. Heat Mass Transfer* **6**, 903–914 (1963).
6. H. F. POPPENDIEK, Turbulent liquid-metal heat transfer in channels, *Nucl. Sci. Engng* **5**, 390–404 (1959).
7. R. A. SEBAN, Heat transfer to a fluid flowing turbulently between parallel walls with asymmetric wall temperatures, *Trans. Am. Soc. Mech. Engrs* **72**, 789–795 (1950).
8. W. M. KAYS and E. Y. LEUNG, Heat transfer in annular passages – hydrodynamically developed turbulent flow with arbitrarily prescribed heat flux, *Int. J. Heat Mass Transfer* **6**, 537–557 (1963).

TRANSFERT THERMIQUE DANS LA RÉGION D'ENTRÉE POUR UN ÉCOULEMENT TURBULENT ÉTABLI ENTRE DEUX PLANS PARALLÈLES

Résumé—On présente une analyse du transfert thermique local dans la région d'entrée pour un profil de vitesse complètement établi à différents nombres de Reynolds et de Prandtl. Des méthodes aux différences sont utilisées pour résoudre l'équation d'énergie. La vitesse et les diffusivités par turbulence sont décrites par des formules semiempiriques. La géométrie est simple: écoulement entre plans parallèles, l'un est thermiquement isolé, l'autre est traversé par un flux thermique connu. Certaines comparaisons avec des calculs sur l'écoulement laminaire sont faites pour éprouver la sureté de la méthode numérique. Ceci fournit aussi une estimation de l'influence de la turbulence dans des cas variés et donne une possibilité de juger la validité des approximations. On donne enfin une comparaison avec des expressions empiriques existantes pour les transfert thermique établi.

WÄRMEÜBERTRAGUNG IM EINLAUF VON PARALLELEN PLATTEN BEI VOLL AUSGEBILDETER TURBULENTER STRÖMUNG

Zusammenfassung—Es wird eine Berechnung des örtlichen Wärmeübergangs in einem thermischen Einlaufgebiet bei voll ausgebildetem turbulentem Geschwindigkeitsprofil für verschiedene Reynolds- und

Prandtl-Zahlen durchgeführt. Die Energiegleichung wird mit Hilfe von Differenzen-Methoden gelöst. Geschwindigkeit und Wirbelleitfähigkeiten werden durch halbempirische Beziehungen beschrieben. Die geometrische Anordnung ist einfach: Strömung zwischen parallelen Platten, wovon eine thermisch isoliert ist, während durch die andere ein bekannter Wärmestrom fliesst. Um die Zuverlässigkeit der numerischen Methode nachzuprüfen, werden bestimmte Vergleiche mit Rechnungen für laminare Strömung gemacht. Daraus ergibt sich auch eine Abschätzung über den Einfluss der Turbulenz in verschiedenen Fällen und die Möglichkeit, die Gültigkeit der Näherungen zu beurteilen. Schliesslich wird ein Vergleich mit verfügbaren empirischen Ausdrücken für voll ausgebildeten Wärmeübergang durchgeführt.

ТЕПЛООБМЕН ВО ВХОДНОМ УЧАСТКЕ ПОЛНОСТЬЮ РАЗВИТОГО ТУРБУЛЕНТНОГО ПОТОКА МЕЖДУ ПАРАЛЛЕЛЬНЫМИ ПЛАСТИНАМИ

Аннотация—Анализируется локальный теплообмен в тепловом входном участке с полностью развитым профилем скорости турбулентного потока при различных числах Рейнольдса и Прандтля. Для решения уравнения энергии используются разностные методы. Полуэмпирическими соотношениями описываются коэффициенты турбулентной диффузии. Геометрическая картина проста: поток между параллельными пластинами, одна из которых теплоизолирована, а к другой подводится заданный тепловой поток. Проведены определённые сравнения с расчётами по ламинарному течению для проверки надежности численного метода. Это также обеспечивает оценку влияния турбулентности в различных случаях и даёт возможность судить о справедливости приближений. Показано также сравнение с имеющимися эмпирическими выражениями для полностью развитого теплообмена.